hep-th/9409185
MPI-PhT/94-64
September 1994

A REMARK ON THE INSTABILITY OF THE BARTNIK-McKINNON SOLUTIONS

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ABSTRACT

The aim of the present letter is to critically review the stability of the Bartnik-McKinnon solutions of the Einstein-Yang-Mills theory. The stability question was already studied by several authors, but there seems to be some confusion about the nature and the number of unstable modes. We suggest to distinguish two different kind of instabilities, which we call 'gravitational' respectively 'sphaleron' instabilities. We claim that the n^{th} Bartnik-McKinnon solution has exactly $2n$ unstable modes, n of either type.

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The discovery of a discrete family of globally regular, static spherically symmetric solutions of Einstein-Yang-Mills (EYM) theory by Bartnik and McKinnon [1] gives rise to the question of their physical interpretation.

It was argued [2, 3] that they are gravitational analogues of the electroweak sphalerons [4]. One of the key points of this interpretation is the presence of unstable mode(s) in the fluctuation spectrum around these solutions [5, 6, 7, 8, 9]. Since the solutions are based on a very special ansatz for the fields one may study as well fluctuations within this ansatz as more general ones. An analysis of fluctuations of the first type was performed in [5, 6, 7], with the result that n negative modes were found for the n^{th} Bartnik-McKinnon (BMK) solution. Inspired by this 'empirical' observation Sudarsky and Wald [3] forwarded an argument that there ought to be just that number of unstable modes of the solutions due to their nature as 'sphalerons' interpolating between topologically inequivalent vacua. In addition they suggested the existence of an additional instability with respect to a (nonlinear) rescaling of the solutions. Thus they claimed that the n^{th} BMK solution has in fact $n+1$ unstable directions. This number of negative modes was 'confirmed' in ref. [10]. Furthermore an analytical proof of the instability of the BMK solutions under spherically symmetric perturbations of the second kind was given in ref. [8, 9].

The aim of the present letter is to critically review these results and clarify some confusing points. In fact, our analysis reveals that the n^{th} solution has n unstable modes of the first type, which may be called 'gravitational' instabilities since they have no analogue for the flat-space sphalerons of the YM-Higgs theory. These are the instabilities discussed in [5, 6, 7]. At least for the lowest member of the BMK family this instability is related to the scaling instability considered by Sudarsky and Wald. In addition there are n unstable modes of the second type, which may be called 'sphaleron' instabilities, because they correspond to the unstable mode found for the YM-Higgs sphaleron [11]. This type of instability of the BMK solutions was discussed in [2, 8, 9]. Altogether we find that the n^{th} BMK solution has $2n$ negative modes within the most general spherically symmetric ansatz.

For our analysis we will essentially adopt the notations of ref. [12]. Due to the spherical symmetry of the solutions the space-time manifold splits into a product $M_2 \times S^2$. The dynamics of the theory can be 'dimensionally reduced' to a 2-dimensional theory on M_2. The line element decomposes as $ds^2 = ds_2^2 + r^2 d\Omega^2$, where $d\Omega^2$ is the invariant line element of the unit 2-sphere. For the metric on M_2 we choose the parametrization

$$ds_2^2 = A^2(t,r)\mu(t,r)dt^2 - \frac{dr^2}{\mu(t,r)} \,. \tag{1}$$

The most general static, spherically symmetric ansatz for the *SU(2)* Yang-Mills field W_μ^a can be written (in the Abelian gauge) as

$$\begin{aligned} W_t^a &= (0,0,A_0)\,, \qquad & W_\theta^a &= (\phi_1,\phi_2,0) \\ W_r^a &= (0,0,A_1)\,, \qquad & W_\varphi^a &= (-\phi_2\sin\theta, \phi_1\sin\theta, \cos\theta)\,. \end{aligned} \tag{2}$$

This ansatz (2) is form invariant under gauge transformations around the third isoaxis, with A_α transforming as a $U(1)$ gauge field on the reduced space-time M_2, whereas $\phi = \phi_1 + i\phi_2$ is a scalar of charge one with the covariant derivative $D_\alpha\phi = \partial_\alpha\phi - iA_\alpha\phi$. With respect to this $U(1)$ one may define the 'charge conjugation' $\phi \to \overline{\phi}$, $A_\alpha \to -A_\alpha$. The reduced EYM action is $S = S_G + S_{YM}$ with ($F_{\alpha\beta}$ denotes the field strength of A_α)

$$S_G = -\frac{1}{2}\int drdtA(\mu + r\mu' - 1) \tag{3}$$

and

$$S_{YM} = -\int drdtA\Big[\frac{r^2}{4}F^{\alpha\beta}F_{\alpha\beta} - \overline{D^\alpha\phi}D_\alpha\phi + \frac{1}{2r^2}(|\phi|^2 - 1)^2\Big]\ . \tag{4}$$

in units where the Newton constant G and the gauge coupling g are set to one. Choosing the gauge $A_0 = 0$ the action S_{YM} can be written more explicitly

$$S_{YM} = -\int drdtA\Big[-\frac{r^2}{2A^2}\dot{A}_1^2 - \frac{1}{A^2\mu}|\dot{\phi}|^2 + \mu|\phi' - iA_1\phi|^2 + \frac{1}{2r^2}(|\phi|^2 - 1)^2\Big]\ , \tag{5}$$

(where a prime denotes $\frac{d}{dr}$ and a dot $\frac{d}{dt}$).

Bartnik and McKinnon [1] found numerically the first few members of an infinite family of static solutions of the field equations derived from $S_G + S_{YM}$. The existence of this family was later proved analytically in refs. [13, 12]. These solutions are even under the $U(1)$ charge conjugation ($\phi_2 = A_1 = 0$) and can be labelled by the number of zeros of the only remaining component $\phi_1 \equiv W$ of the YM field. In order to analyse their stability under small perturbations we have to consider the spectrum of (harmonically) time dependent perturbations in the background of the BMK solutions. The existence of solutions of the linearized field equations corresponding to imaginary frequency indicates the instability of the background solution leading to an exponential growth of the perturbation in time.

For perturbations of the type

$$\begin{array}{rl} \phi_1 \to W(r) + \varphi(r)\mathrm{e}^{i\omega t} & \qquad A \to A(r) + a(r)e^{i\omega t} \\ \phi_2 \to \psi(r)\mathrm{e}^{i\omega t} & \qquad \mu \to \mu(r) + \kappa(r)e^{i\omega t} \\ A_1 \to a_1 e^{i\omega t} & \end{array} \tag{6}$$

the linearized field equations are

$$\begin{aligned} A\mu\Big(W'\psi - W(\psi' - a_1 W)\Big) &= \frac{\omega^2 r^2}{2A}a_1 \\ -(A\mu\varphi')' - \Big((A\kappa + \mu a)W'\Big)' + A\frac{3W^2 - 1}{r^2}\varphi + \frac{W(W^2 - 1)}{r^2}a &= \frac{\omega^2}{A\mu}\varphi \\ -\Big(A\mu(\psi' - a_1 W)\Big)' + A\mu W' a_1 + A\frac{W^2 - 1}{r^2}\psi &= \frac{\omega^2}{A\mu}\psi \\ ra' - 4AW'\varphi' - 2W'^2 a &= 0 \end{aligned}$$

$$(r\kappa)' + 2W'^2\kappa + 4\mu W'\varphi' + \frac{4W(W^2-1)}{r^2}\varphi \;=\; 0\,, \tag{7}$$

with W, μ and A the background solutions. The last equation can be integrated imposing the boundary condition $\kappa(\infty) = 0$ with the result

$$\kappa = -\frac{4\mu W'}{r}\varphi\,. \tag{8}$$

Putting back this expression for κ into the equation for φ allows to eliminate the gravitational degrees of freedom κ and a from the equations for the YM field. Furthermore the equations for the even and odd sectors under the $U(1)$ charge conjugation decouple. One obtains

$$-\,(A\mu\varphi')' + A\frac{3W^2-1}{r^2}\varphi + \Big(\frac{4A\mu W'^2}{r}\Big)'\varphi = \frac{\omega^2}{A\mu}\varphi, \tag{9}$$

and

$$\begin{aligned} A\mu\Big(W'\psi - W(\psi' - a_1 W)\Big) \;&=\; \frac{\omega^2 r^2}{2A}a_1 \\ -\Big(A\mu(\psi' - a_1 W)\Big)' + A\mu W' a_1 + A\frac{W^2-1}{r^2}\psi \;&=\; \frac{\omega^2}{A\mu}\psi\,. \end{aligned} \tag{10}$$

The φ-sector was analyzed in the earlier works [5, 6, 7] with the result that n negative modes for the n^{th} BMK solution were found. Up to now this coincidence was considered as a curiosity for which no explanation was offered. Although we also cannot prove this coincidence, we want to give some 'explanation' for it. One may add a mass term

$$S_m = -\frac{\alpha^2}{4}\int dr dt A\Big((\phi_1+1)^2 + \phi_2^2\Big) \tag{11}$$

to the action for the YM field. Such a term results from the coupling of a Higgs field (doublet) in the limit of infinite Higgs mass. The resulting theory was studied in [14]. As the mass α is varied one finds one-parameter families of solutions tending to the BMK solutions in the limit $\alpha \to 0$. These families have the property that they interpolate continuously between the n^{th} and $n{+}1^{\mathrm{st}}$ BMK solution as α increases from zero to some maximal value $\alpha^{(n)}_{\max}$ and then runs back to zero. The points $\alpha = \alpha^{(n)}_{\max}$ are bifurcation points at which the branch starting from the n^{th} BMK solution and the one starting from the $n+1^{\mathrm{st}}$ BMK solution merge. It is well known that at such bifurcation points the number of unstable modes changes generically by one. Thus starting with the stable trivial solution (Minkowski space, $W \equiv 1$) we end up with one unstable mode for the first non-trivial BMK solution and so on.

A further remark may be at order. Although it is true that for the lowest BMK solution the scaling of the solution proposed in [3] yields an unstable direction this is not generally so for the higher ($n > 2$) BMK solutions [7].

Next let us turn to the $U(1)$ charge conjugation odd ψ sector. As already mentioned an analytical proof of the existence of at least one unstable mode of this type has been given in [8, 9]. We have performed a numerical analysis of the fluctuation spectrum for the first few members of the BMK family. We find that the n^{th} solution has n negative modes also in the ψ channel. In fact, this is what was suggested by Sudarsky and Wald [3], ignoring that they interpreted these instabilities as the ones found in the φ sector.

Hence altogether we claim that the n^{th} BMK solution has $2n$ unstable modes with respect to spherically symmetric perturbations.

For comparison we have collected in Tables 1 and 2 our numerical values for the energies $E = \omega^2$ of the negative modes of the first three BMK solutions in the φ resp. ψ sectors.

$N = 1$	$N = 2$	$N = 3$
$E_1 = -0.0525$	$E_1 = -0.0410$	$E_1 = -0.0339$
	$E_2 = -0.0078$	$E_2 = -0.0045$
		$E_3 = -0.0006$

Tab 1. Boundstate energies for the $N = 1, 2, 3$ BMK solutions, (φ sector).

$N = 1$	$N = 2$	$N = 3$
$E_1 = -0.0619$	$E_1 = -0.0360$	$E_1 = -0.0346$
	$E_2 = -0.0105$	$E_2 = -0.0037$
		$E_3 = -0.0009$

Tab 2. Boundstate energies for the $N = 1, 2, 3$ BMK solutions, (ψ sector).

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